A Towed, Hypersonic Balloon

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A mathematical analysis is presented for the wave drag and axisymmetric distortion, under the assumption of infinitesimal strains, of an orginally spherical, inflated balloon towed in an inviscid, hypersonic flow. The distortion is such that longitudinal creases develop at the leading edge (towing point) and extend rearwards, the solution in the creased region being matched with that appropriate to the remainder of the surface. The relevant matching conditions are selected from nonlinear membrane theory without the necessity of a full nonlinear analysis. The ratio of the drag of the balloon to that of a rigid sphere of the same undistorted size is calculated as a function of the ratio of stagnation pressure behind a normal shock wave to the internal inflation pressure. For low values of this pressure ratio (<0.8), the drag of the balloon is slightly larger than that of the sphere. A comparison has been made with other hypersonic decelerators. The method is suitable for extension in order to cover more complicated shapes.

Nomenclature

a	= radius of the undistorted balloon
A, B, B', C, D	= parameters defined after Eqs. (3)
C_D	= drag coefficient based on surface area
D_{balloon}	= wave drag of balloon
D_{sphere}	= wave drag of sphere $= \pi a^2 p_0/2$
E	= elastic modulus
h	= membrane thickness
N_θ	= circumferential load/unit width of surface
N_ϕ	= longitudinal load/unit width of surface
p_i	= balloon inflation pressure
p_n	= resultant outward pressure
p_0	= stagnation pressure behind a normal shock wave
P	= p_0/p_i
q_∞	= freestream dynamic pressure
r'	= distance of longitudinal meridian from axis of symmetry
r	= r'/a
r_1	= longitudinal radius of curvature
r_c	= r at start of creased region
s	= distance along the longitudinal meridian
T	= longitudinal tension/unit radian of circumferential rotation
ϵ	= $p_n a/(Eh)$
η_θ	= $N_\theta/(p_i a/2)$
η_ϕ	= $N_\phi/(p_i a/2)$
ϕ	= angle between inward normal to the surface and the flow direction
ϕ_c	= ϕ at start of creased region
ϕ_L	= ϕ at leading edge ($r = 0$)
τ	= $2T/(p_i a^2)$

THE object of this paper is to present an analysis of the axisymmetric distortion under infinitesimal strains, and resulting change in wave drag, of an originally spherical inflated membrane (which has no bending stiffness) towed in inviscid, hypersonic flow. This analysis has been motivated by the experimental investigation of inflatable, reentry decelerators. Recent associated theoretical work has been published by Daskin and Feldman[1] on two-dimensional, hypersonic sails, and Boyd[2] on the drag of drogue and umbrella-shaped, hypersonic parachutes. The weight advantage of such devices is obvious; the balloon would seem rather more attractive inasmuch as it does not require rigid support rings, and hence should occupy a smaller packed volume.

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Using the modified Newtonian hypersonic pressure approximation, the resultant outward normal pressure is given by

$$p_n = p_i - p_0 \cos^2\varphi \qquad \text{for} \qquad 0 \leq \varphi \leq \pi/2 \qquad (1a)$$

$$p_n = p_i \qquad \text{for} \qquad \pi/2 \leq \varphi \leq \pi \qquad (1b)$$

where p_i is the constant internal pressure, p_0 is the stagnation pressure behind a normal shock wave, ϕ is the angle betwen the inward normal to the surface and the flow direction. The point of support becomes the leading edge and the meridian curve develops into a shape similar to that shown in Fig. 1. In this configuration, the surface can be divided into two regions: a creased region in which longitudinal creases run from the leading edge to a point where the circumferential tension in the surface becomes positive, and the remaining balloon surface, characterized by a positive circumferential tension. In this analysis the strains in the material will be considered to be small enough to induce no secondary surface loading effects. (This assumption is consistent with the properties of the materials likely to be used for such a device.) The problem is approached by writing down the equations of equilibrium appropriate to each region and solving, subject to the matching conditions at the join. These boundary conditions are continuity of deflection, slope, and loading, together with the requirement that the new strained meridian length is compatible with the originally circular meridian length under the action of the induced stresses.

In the creased region, a description of the geometry of the creasing is avoided by simply using the equation of longitudinal equilibrium for no circumferential stress. Thus, the physics of the longitudinal creases is not described in this analysis, since the material is assumed to be capable of folding circumferentially into very small, regular creases to produce a surface of revolution formed by revolving the generator (longitudinal meridian), shown in Fig. 1, about the axis of symmetry. The load is then completely carried by longitudinal fibers and the differential equation is the same as that used by Boyd[2] for the solution of the creased parachute problem:

$$(T/r')(d\varphi/ds) = p_n \qquad (2)$$

where T is the longitudinal tension per unit radian of circumferential rotation, s is a length measured along the longitudinal generator, and r' is the distance from the axis of symmetry. T is a constant and can be determined from equilibrium at the point of initial wrinkling. Since p_n is given by

Eq. (1) and $dr' = ds \cos\phi$, the solution to Eq. (2) for the shape of the creased meridian may be written after suitable nondimensionlization, using the radius a of the undistorted balloon in the following forms, $\phi_c \leq \pi/2$ (the subscript c refers to the conditions at the join):

$$P = p_0/p_i < 1 \qquad \sin\varphi = [(1 - P)/P]^{1/2} \tan\{A - B(r_c^2 - r^2)\} \qquad (3a)$$

$$P > 1 \qquad \sin\varphi = \left(\frac{P-1}{P}\right)^{1/2} \frac{Ce^{B'(r_c^2 - r^2)} - D}{Ce^{B'(r_c^2 - r^2)} + D} \qquad (3b)$$

where

$$\tan A = \left(\frac{P}{1-P}\right)^{1/2} \sin\varphi_c \qquad B = \left(\frac{1-P}{P}\right)^{1/2} \frac{P}{\tau}$$

$$B' = 2\left(\frac{P-1}{P}\right)^{1/2} \frac{P}{\tau} \qquad C = \left(\frac{P-1}{P}\right)^{1/2} + \sin\varphi_c$$

$$D = \left(\frac{P-1}{P}\right)^{1/2} - \sin\varphi_c \qquad \tau = \frac{2T}{(p_i a^2)} \qquad r = \frac{r'}{a}$$

Instead of integrating the pressure forces over the balloon surface to find the wave drag, it is simpler to equate the component of membrane forces in the axial direction at the leading edge to the drag, i.e.,

$$D_{\text{balloon}} = 2\pi T \sin\varphi_L$$

where the subscript L refers to the leading edge, i.e., at $r = 0$. (In fact, in the actual construction of such a balloon, in order to keep the stresses below an acceptable limit, there would be a lower bound on r which would be a function of the material properties.) Since the wave drag of the undistorted sphere is $\pi a^2 p_0/2$, the drag ratio is

$$D_{\text{balloon}}/D_{\text{sphere}} = (2\tau/P) \sin\varphi_L \qquad (4)$$

Thus, once the matching has been achieved and τ has been found, the appropriate Eq. (3) will enable the drag ratio to be found by Eq. (4).

It is now necessary to investigate the uncreased region. The usual equations of equilibrium of an axisymmetric membrane under normal loading are

$$(d/d\varphi)(r'N_\varphi) - N_\theta r_1 \cos\varphi = 0 \qquad (5a)$$

$$N_\varphi/r_1 + (N_\theta/r') \sin\varphi = p_n \qquad (5b)$$

where N_ϕ and N_θ are the longitudinal and circumferential loads per unit width of surface whose longitudinal radius of curvature is r_1. The other symbols are as previously defined.

If the strains are infinitesimal, then for a spherical region of radius a,

$$r_1 = a \qquad r' = a \sin\phi$$

Equations (5) may be solved immediately to give the stresses in an inelastic shell subject to Eq. (1a):

$$\eta_\varphi = \frac{N_\varphi}{(p_i a/2)} = 1 + \frac{P \cos^4\varphi}{2 \sin^2\varphi} \qquad (6)$$

$$\eta_\theta = \frac{N_\theta}{(p_i a/2)} = 1 - \frac{P \cos^2\varphi(4 - 3\cos^2\varphi)}{2 \sin^2\varphi} \qquad (7)$$

where the constant of integration has been selected to make $\eta_\phi = \eta_\theta = 1$ at $\phi = \pi/2$ (this condition holds for $\pi/2 \leq \phi \leq \pi$). Equation (7) immediately defines $\phi_c(\eta_\theta = 0)$, and the matching of the solutions would then require the following boundary conditions, which have been discussed already, at $r_c = \sin\phi_c$:

$$\varphi_c(\text{creased region}) = \varphi_c(\eta_\theta = 0) \qquad (8a)$$

$$\tau = \eta_{\varphi c} \sin\varphi_c \qquad (8b)$$

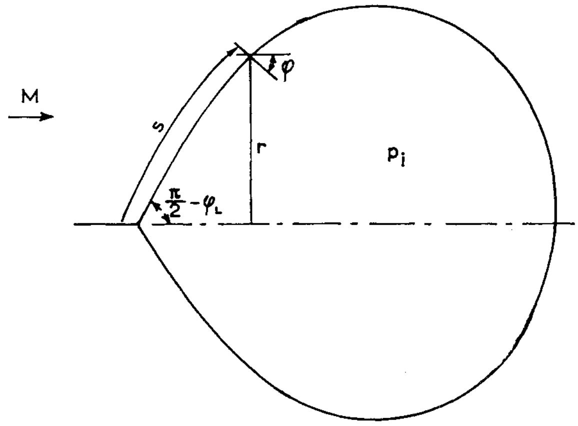

Fig. 1 The towed balloon.

$$\int_{\text{creased region}} ds = \int_0^{\sin\varphi_c} \frac{dr}{\cos\varphi} = \varphi_c \qquad (8c)$$

Of these, only two may be satisfied simultaneously. If, for example, Eqs. (8a) and (8b) were used, then the right-hand side of Eq. (8c) can be shown to be less than the left-hand side, the difference becoming larger as P increases. This behavior is not unexpected since linear membrane theory, as used previously, is usually unable to deal with all the boundary conditions. Bending stresses or nonlinear membrane theory are normally introduced in boundary-layer regions to deal with continuity of deflections. The solution, as outlined previously, is therefore only valid for $P \to 0$ and has limited use. To extend the scope of the analysis, nonlinear membrane theory has to be used. A recent perturbation solution in terms of a small parameter $\epsilon = p_n a/(Eh)$ (where E is the elastic modulus, h is the membrane thickness, and the other symbols have been previously defined) for deep membranes of revolution under axisymmetric loading has been given by Rossettos[3] based on the equations of Sanders[4] for small strains and moderately small rotations, and may be applied directly to this problem. Rossettos' treatment gives the quite general result that, in the neighborhood of boundaries where linear theory inadequately describes the membrane behavior, exponentially decaying boundary-layer regions form, having corrections to the linear result of zeroth order in N_θ and curvature (but not in slope), with $0(\epsilon^{1/2})$ corrections to the other relevant variables.† This may be stated simply as follows: to 0(1), changes in longitudinal curvature cause compensatory changes in N_θ such that equilibrium [Eqs. (5)] is still preserved without alteration of the other variables. For purposes of obtaining simple answers, only terms of 0(1) are retained from the non-linear theory; this analysis then retains its simplicity (for ϕ_c not too small, but this corresponds to $P \to 0$ for which a solution is already known), since Eq. (7), for N_θ, is disregarded because the nonlinear correction is absent, and Eqs. (8), which still hold to 0(1), are now satisfied [i.e., a value of ϕ_c must be found, independently of Eq. (7), to satisfy Eqs. (8)]. Hence, these equations may be interpreted as defining the boundary between creased and uncreased regions. It is easy to see that, chosen in this way, $\phi_c > \phi_c[\eta_\theta = 0$, from Eq. (7)]; hence, the zeroth-order nonlinear correction to Eq. (7) must be negative, decaying exponentially with an exponent proportional to $(\epsilon^{1/2})$ (Ref. 3) for $\phi > \phi_c$, making the circumferential stress zero at ϕ_c. It also follows that, in the neighborhood of ϕ_c, the curvature must increase and, in fact, at ϕ_c it can be shown to be equal to the curvature in the creased region.

Thus a solution to 0(1) has been found for the shape and wave drag of a towed balloon in hypersonic flow. It is not

† Equations (29) and (37) of Rossettos's paper.[3]

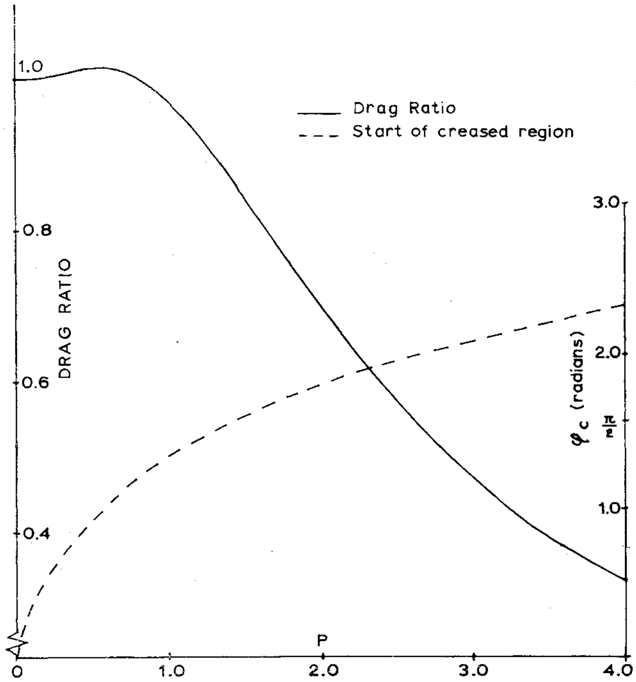

Fig. 2 Variation of drag ratio and area of creased region with *P*.

possible to write the solution explicitly in closed form because of the integral in Eq. (8c). However, for a given P, this integral may be evaluated numerically in an iterative procedure to find ϕ_c; then Eq. (6) gives η_θ, and Eq. (8b) gives τ and the drag ratio from Eq. (4). The results, for $P \leq 4$ (the probable extent of the working range of the balloon as a decelerator), are presented in Table 1, and Fig. 2, with the corresponding values of ϕ_c for interest. Using the method outlined to find ϕ_c, the longitudinal creases may continue into the region $\phi > \pi/2$ for P sufficiently large (>1.45, approximately), in which case it is necessary to modify the analysis somewhat. Equation (3b) applies until $\phi = \pi/2$,

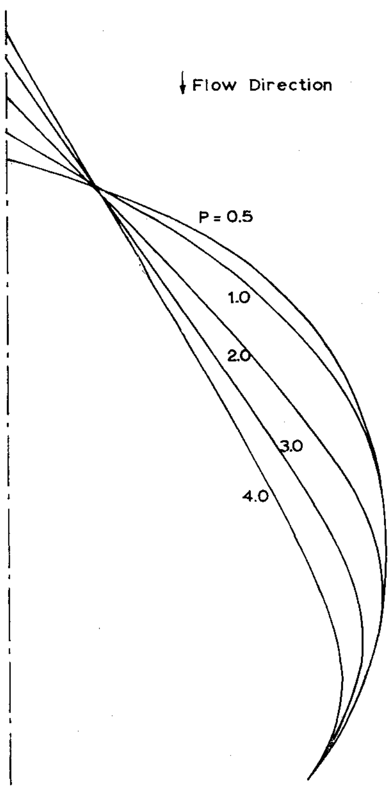

Fig. 3 Balloon distortion (meridian shapes).

Table 1 Drag ratios and wrinkling position of towed, hypersonic balloon

$P = p_0/p_i$	Drag ratio = $\frac{\text{Drag (balloon)}}{\text{Drag (sphere)}}$	ϕ_c, rad
0	0	0
0.2	1.00	0.53
0.4	1.01	0.80
0.6	1.02	1.02
0.8	1.00	1.19
1.0	0.97	1.32
1.5	0.85	1.57
2.0	0.70	1.80
2.5	0.58	1.95
3.0	0.48	2.07
3.5	0.39	2.20
4.0	0.34	2.25

then Eq. (1b) in Eq. (2) gives (using $\phi = \phi_c$ at $r = \sin\phi_c$, and $\tau = \sin\phi_c$)

$$\sin\varphi(\varphi \geq \pi/2) = r^2/\sin\varphi_c$$

Thus, at $\phi = \pi/2$, $r = (\sin\phi_c)^{1/2}$, whence Eq. (3b) should be reinterpreted with r_c and ϕ_c referring to conditions at $\phi = \pi/2$ with $\tau = 1$. Figure 3 shows some typical meridian shapes.

It is interesting to note that the results predict a marginal improvement in balloon drag for $P < 0.8$ (approximately) and, even when the balloon is pressurized by the stagnation pressure behind the normal shock (by having an opening at the leading edge: the self-inflated or ramair configuration), $P = 1$, the drag is still quite high. A comparison with Boyd's[2] uncorrected parachute results may be made since, in the notation of that paper, $q_\infty C_D/p_0 = \frac{1}{8}$ for a rigid sphere and a little less for the self-inflatable balloon. (Here, q_∞ is the freestream dynamic pressure and C_D is the drag coefficient based on the total surface area.) This parameter represents the ratio of drag to surface area for a given flow and, therefore, may be regarded as a measure of the decelerator's efficiency. Referring to Fig. 5 of Boyd's paper, it may be observed that, for a parachute whose trailing edge angle is fixed at 90°, the value of $q_\infty C_D/p_0$ may vary between 0 and 1 as the leading edge angle of the parachute varies between 0° and 90°. Therefore, it would appear at first sight that the balloon is a rather less efficient drag device. However, as stated in the conclusion of Boyd's paper, the leading-edge angles are restricted by maximum tension considerations, which limit the magnitude of $q_\infty C_D/p_0$ to values not very different from those of the balloon. For example, using Boyd's illustration, the design of a device offering 20,000 lb drag at $M_\infty = 10$ at an altitude of 100,000 ft requires (after the selection of a leading edge angle of 9°) an uncorrected parachute whose area is approximately 40 ft^2, and a maximum diameter of approximately 4 ft. The self-inflated balloon, under the same flow conditions, would need a surface area of approximately 50 ft^2 with a diameter of 4 ft; it should be remembered that the parachute requires a rigid trailing edge support (and an axial member if the trailing edge angle is less than 90°) to remove the tension load in the membrane.

All of these results, of course, assume that the balloon is towed far enough behind the vehicle for the latter's base flow not to appreciably affect the hypersonic pressure distribution. Although some experimental work has been done on the balloon as a towed decelerator, none is readily quantitatively comparable with the results of this paper. However, it has been previously stated,[5] and confirmed here for the sphere, that self-inflatable decelerators have approximately the same drag as their solid counterparts. Finally, in any practical application of the results, it might be necessary to calculate volume changes in order to predict the variation in internal pressure; these may be readily calculated from the present analysis. Further, it is quite possible to extend this type of approach to study more complicated balloon shapes if the need arises.

References

[1] Daskin, W. and Feldman, L., "The characteristics of two-dimensional sails in hypersonic flow," J. Aeronaut. Sci. **25**, 53–55 (1958).

[2] Boyd, E. A., "Aerodynamic characteristics of a hypersonic parachute," College of Aeronautics, Rept. 152 (1961).

[3] Rossettos, J. N., "Nonlinear membrane solutions for symmetrically loaded deep membranes of revolution," NASA Langley Research Center, TN D-3297 (1966).

[4] Sanders, J. L., "Nonlinear theories for thin shells," Quart. Appl. Math. **21**, 21–36 (1963).

[5] McShera, J. T., "Aerodynamic drag and stability characteristics of towed inflatable decelerators at supersonic speeds," NASA Langley Research Center, TN D-1601 (1963).

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Systematic Determination of Simplified Gain Scheduling Programs

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Given the control-loop gain settings for best handling qualities at each individual flight condition of interest, this paper presents a direct procedure for determining which simplified gain programs would yield the best performance for their level of complexity, and the approximate numerical values for these simplified gain schedules. Since the approach is suitable for digital mechanization, this phase of control system design can be largely automated.

Nomenclature

K,x,y,w	=	variables
Z	=	the numerical value of a function
i,j,g	=	running indexes
I,J,G	=	upper limits on indexes (i), (j), and (g), respectively
T_{ij}	=	summation of (K) values over all indexes but (i) and (j); (T) would imply the summation over all (K) values with no exceptions, etc.
M_i	=	measure of the importance of the ith function
SS	=	summation of (K^2) values over all indexes
KP	=	roll rate feedback to aileron
KR	=	yaw rate feedback to rudder
KCF	=	stick crossfeed of roll command into rudder

Introduction

GIVEN a set of flight condition variables (such as Mach number, angle of attack, dynamic pressure, center of gravity location, etc.), control-loop gain settings yielding the best handling qualities at each individual flight condition can be directly selected. However, the manner in which the gain scheduling should be simplified to achieve a practical level of airborne mechanization complexity with a minimum degradation in handling qualities is not so obvious. The technique presented in this article offers a direct method of doing this. The first section to follow presents a standard form for a functional relationship with two independent variables. Continuing with the two-variable case, the second section derives measures of the importance of each functional component to the dependent variable value, and develops expressions for computing these measures. The third section extends the two-variable case results to three variables, making the extension to four or more variables evident. Next, the logic for selecting control configurations from the computed data and an example application are presented. Finally, possible selection logic refinements and the manner in which numerical gain approximations are computed are discussed, in addition to summarizing the approach.

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Standard Functional Form

Given a dependent variable (K), which is a function of two independent variables (x) and (y), the functional relationship can always be expressed in two alternate forms as follows:

$$K = f(x,y) \qquad \text{or} \qquad K = \langle K\rangle + f(x) + f(y) + f(x,y)$$

For any set of (x) and (y) values, the values of the functions on the right-hand side of the latter expression can always be evaluated. For example, define the symbols (Z), (Z_i), (Z_j), and (Z_{ij}) as follows: Z = the value of the const $\langle K\rangle$; Z_i = the value of $f(x)$ for the ith value of (x); Z_j = the value of $f(y)$ for the jth value of (y); Z_{ij} = the value of $f(x,y)$ for the ith value of (x) and the jth value of (y). Then the latter expression becomes

$$K_{ij} = Z + Z_i + Z_j + Z_{ij}$$

By definition, the summations of the functions (Z_i), (Z_j), and (Z_{ij}) over their respective indexes are to be zero, i.e.,

$$\sum_i Z_i \equiv \sum_j Z_j \equiv \sum_i Z_{ij} \equiv \sum_j Z_{ij} \equiv 0$$

Symbolizing the maximum value of the (i) index as (I), and the maximum value of the (j) index as (J), the (Z_i) values can be derived as follows:

$$\sum_i K_{ij} = \sum_i Z + \sum_i Z_i + \sum_i Z_j + \sum_i Z_{ij} = IZ + IZ_j$$

$$Z_j = \left(\sum_i K_{ij} \Big/ I\right) - Z$$